A Bi-Hamiltonian Theory for Stationary KdV Flows and their Separability

Gregorio Falqui[1], Franco Magri[2],

Marco Pedroni[3] and Jorge P. Zubelli[4]

[1] SISSA, Via Beirut 2/4, I-34014 Trieste, Italy
E–mail: falqui@sissa.it

[2] Dipartimento di Matematica e Applicazioni
Università di Milano–Bicocca
Via degli Arcimboldi 8, I-20126 Milano, Italy
E–mail: magri@vmimat.mat.unimi.it

[3] Dipartimento di Matematica, Università di Genova
Via Dodecaneso 35, I-16146 Genova, Italy
E–mail: pedroni@dima.unige.it

[4] IMPA
Est. D. Castorina 110, Rio de Janeiro, RJ 22460, Brazil
E–mail: zubelli@impa.br

Abstract

We present a fairly new and comprehensive approach to the study of stationary flows of the Korteweg-de Vries hierarchy. They are obtained by means of a double restriction process from a dynamical system in an infinite number of variables. This process naturally provides us with a Lax representation of the flows, which is used to find their bi-Hamiltonian formulation. Then we prove the separability of these flows making use of their bi-Hamiltonian structure, and we show that the variables of separation are supplied by the Poisson pair.

1 Introduction

In this paper we present a comprehensive approach to the stationary flows of the Korteweg–de Vries (KdV) equation. This is a quite classical subject, and lies at the very heart of the modern theory of integrable systems in finite and infinite dimensions. Its formulation can be traced back at least to [6, 23]. Then, together with its generalizations, it has been subject of intensive studies. See, for example, [7, 12, 13, 14, 22, 28] and references therein. It was realized that such systems were among the prototypes of the class of Algebraically Completely Integrable Hamiltonian Systems [2, 29], and, in particular, it has been proven that the classical Jacobi formulas for projective embeddings of hyperelliptic Jacobian varieties found a very natural realization in such problems.

Hamiltonian aspects of those flows were studied in a number of papers. To cite a few [6, 14, 23]. Two main approaches emerged. In the first one, a Hamiltonian structure was given to the stationary KdV flows, looking at their variational properties [6]. Namely, such flows were regarded as classical Euler–Lagrange equations associated to suitable Lagrangian and Hamiltonian densities. In the second one (see, e.g., [13]), a set of suitable canonical coordinates was introduced on the stationary manifolds, somehow dictated by the algebro–geometrical structure of the problem, and the fact that the flows were indeed Hamiltonian was verified at a later stage, via direct computation. In the same way, a set of action–angle variables was found.

After the discovery of the bi-Hamiltonian structure of the $1+1$ dimensional KdV hierarchy, the obvious problem of finding a corresponding bi-Hamiltonian structure for its stationary flows was studied. A solution for this was found [4, 32, 37] with an ingenious study mainly based on the recursion relations found by Alber [3], the introduction of special sets of coordinates, and the use of the Miura transformation.

The aim of this paper is to give a rather new and systematic perspective to this circle of ideas, focusing our attention on the bi-Hamiltonian aspects of the problem. The next section is devoted to a quick description, in a simple example, of the stationary reductions of KdV and to an illustration of the main properties of these systems. Then, we will present the plan and the important points of the paper.

2 A Preliminary View

In this section we recall some known facts about the KdV hierarchy and its stationary reductions, and we present the problems to be tackled in the next sections.

The KdV equation is the most famous example in the class of the so-called integrable nonlinear PDEs. It possesses a number of remarkable properties, in particular:

1. It has an infinite sequence of integrals of motion;
2. It admits a Lax representation;
3. It is a bi-Hamiltonian system;
4. The integrals of motion are the coefficients of a Casimir of the Poisson pencil, so that the KdV equation can be seen as a *Gel'fand–Zakharevich system* (see below).

There are of course relations between these properties. For example, the conserved densities are the residues of the fractional powers of the Lax operator. Also, they can be extracted from the bi-Hamiltonian structure and shown to commute with respect to both Poisson brackets. The

associated vector fields form the KdV hierarchy, whose first members are

$$
\begin{aligned}
&\frac{\partial u}{\partial t_1} = u_x, \qquad \frac{\partial u}{\partial t_3} = \frac{1}{4}(u_{xxx} - 6uu_x) \quad \text{(KdV equation)} \\
&\frac{\partial u}{\partial t_5} = \frac{1}{16}(u_{xxxxx} - 10uu_{xxx} - 20u_x u_{xx} + 30u^2 u_x) \ .
\end{aligned}
\tag{2.1}
$$

The KdV hierarchy can be used to find finite-dimensional reductions for the KdV equation, giving rise to explicit solutions. Indeed, the set of singular points of a (fixed) vector field of the hierarchy is a finite-dimensional manifold which is invariant under the flows of the other vector fields, due to the commutativity property. The (finite-dimensional) systems obtained by restricting the KdV hierarchy to such invariant manifolds are called the *stationary reductions of KdV*.

Let us consider explicitly the reduction KdV_5 corresponding to the third vector field of the hierarchy. The set of zeroes is given by

$$
u_{xxxxx} - 10uu_{xxx} - 20u_x u_{xx} + 30u^2 u_x = 0 \ , \tag{2.2}
$$

and its dimension is 5, since we can use the value of u, u_x, u_{xx}, u_{xxx}, and u_{xxxx} at a fixed point x_0 (i.e., the Cauchy data) as global coordinates. For the sake of simplicity we put

$$
u_0 = u(x_0), \quad u_1 = u_x(x_0), \quad u_2 = u_{xx}(x_0), \quad u_3 = u_{xxx}(x_0), \quad u_4 = u_{xxxx}(x_0) \ . \tag{2.3}
$$

In order to compute the reduced equations of the first flow of (2.1), we have to derive it with respect to x, and to use the constraint (2.2) and its differential consequences to eliminate all the derivatives of order higher than 4. We obtain the equations

$$
\begin{aligned}
&\frac{\partial u_0}{\partial t_1} = u_1, \quad \frac{\partial u_1}{\partial t_1} = u_2, \quad \frac{\partial u_2}{\partial t_1} = u_3, \quad \frac{\partial u_3}{\partial t_1} = u_4, \\
&\frac{\partial u_4}{\partial t_1} = 10u_0 u_3 + 20u_1 u_2 - 30{u_0}^2 u_1 \ .
\end{aligned}
\tag{2.4}
$$

In the same way, for the KdV equation we get

$$
\begin{aligned}
\frac{\partial u_0}{\partial t_3} &= \frac{1}{4}(u_3 - 6u_0 u_1) \\
\frac{\partial u_1}{\partial t_3} &= \frac{1}{4}(u_4 - 6u_0 u_2 - 6{u_1}^2) \\
\frac{\partial u_2}{\partial t_3} &= \frac{1}{4}(4u_0 u_3 + 2u_1 u_2 - 30{u_0}^2 u_1) \\
\frac{\partial u_3}{\partial t_3} &= \frac{1}{4}(4u_0 u_4 + 6u_1 u_3 + 2{u_2}^2 - 30{u_0}^2 u_2 - 60u_0 {u_1}^2) \\
\frac{\partial u_4}{\partial t_3} &= \frac{1}{4}(10u_1 u_4 + 10{u_0}^2 u_3 + 10u_2 u_3 - 100u_0 u_1 u_2 - 60{u_1}^3 - 120{u_0}^3 u_1)
\end{aligned}
\tag{2.5}
$$

As far as the restrictions of the other flows are concerned, it can be shown that they are linear combination of (2.4) and (2.5).

It is not surprising that the above mentioned properties of the KdV equation hold also for the KdV_5 system. However, to the best of our knowledge, it has not been made completely clear the

way which these properties pass from the KdV hierarchy to its stationary reductions, especially as far as the bi-Hamiltonian structure is concerned. In any case, one can check that the functions

$$\begin{aligned}
H_0 &= \frac{1}{16}(-2u_2u_4 + 6{u_0}^2u_4 + {u_3}^2 - 12u_0u_1u_3 + 16u_0{u_2}^2 + 12{u_1}^2u_2 - 60{u_0}^3u_2 + 36{u_0}^5) \\
H_1 &= -\frac{1}{4}(2u_0u_4 - 2u_1u_3 + {u_2}^2 - 20{u_0}^2u_2 + 15{u_0}^4) \\
H_2 &= u_4 - 10u_0u_2 - 5{u_1}^2 + 10{u_0}^3
\end{aligned} \tag{2.6}$$

are integrals of motion for (2.4) and (2.5). These systems have also a Lax formulation, i.e., they can be written as

$$\frac{\partial L}{\partial t_i} = [A_i, L], \qquad i = 1, 3, \tag{2.7}$$

where the Lax matrix L depends on a parameter λ, and is given by

$$L = \frac{1}{16}\begin{pmatrix} 4u_1\lambda + u_3 - 6u_0u_1 & 16\lambda^2 - 8u_0\lambda + 6{u_0}^2 - 2u_2 \\ \begin{matrix} 16\lambda^3 + 8u_0\lambda^2 + 2\lambda(u_2 - {u_0}^2) + \\ u_4 - 8u_0u_2 - 6{u_1}^2 + 6{u_0}^3 \end{matrix} & -4u_1\lambda - u_3 + 6u_0u_1 \end{pmatrix}. \tag{2.8}$$

The matrices A_i can be easily contructed from L (see Section 4).

Finally, there are two compatible Poisson structures giving a (bi)-Hamiltonian formulation of the KdV$_5$ systems. The corresponding Poisson tensors are

$$P_0 = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 & -20u_0 \\ 0 & 2 & 0 & 20u_0 & 20u_1 \\ -2 & 0 & -20u_0 & 0 & -140{u_0}^2 - 20u_2 \\ 0 & 20u_0 & -20u_1 & 140{u_0}^2 + 20u_2 & 0 \end{bmatrix}$$

and

$$P_1 = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 3u_0 & 6u_1 \\ -\frac{1}{2} & 0 & -3u_0 & -3u_1 & -4u_2 - 15{u_0}^2 \\ 0 & 3u_0 & 0 & u_2 + 15{u_0}^2 & u_3 + 30u_0u_1 \\ -3u_0 & 3u_1 & -u_2 - 15{u_0}^2 & 0 & \begin{matrix} u_4 - 40u_0u_2 + \\ 30{u_1}^2 - 60{u_0}^3 \end{matrix} \\ -6u_1 & 4u_2 + 15{u_0}^2 & -u_3 - 30u_0u_1 & \begin{matrix} -u_4 + 40u_0u_2 - \\ 30{u_1}^2 + 60{u_0}^3 \end{matrix} & 0 \end{bmatrix}.$$

If we call X_1 and X_3 the vector fields of KdV_5, then the following relations hold:

$$\begin{aligned} P_0 dH_2 &= 0 \\ X_1 = P_0 dH_1 &= P_1 dH_2 \\ X_3 = P_0 dH_0 &= P_1 dH_1 \\ P_1 dH_0 &= 0 \; . \end{aligned} \tag{2.9}$$

They can be collected in the statement that the function $H(\lambda) := H_2\lambda^2 + H_1\lambda + H_0$ is a Casimir of the *Poisson pencil* $P_\lambda := P_1 - \lambda P_0$, that is,

$$P_\lambda dH(\lambda) = 0 \; . \tag{2.10}$$

Therefore, X_1 and X_3 are the bi-Hamiltonian vector fields associated with a polynomial Casimir of a Poisson pencil of maximal rank. In a word, they are Gel'fand–Zakharevich (GZ) systems [19].

The importance of the stationary reductions of the KdV hierarchy (and, more generally, of the stationary reductions of the Gel'fand–Dickey hierarchies) lies in the fact that the reduced equations can be solved by means of the classical method of separation of variables. This was noticed in the early works on the subject. It is also known how to construct the variables of separation starting from the Lax matrix. We will show that the separability of these systems is a particular instance of a general result, which is valid for quite a wide class of bi-Hamiltonian manifolds.

In Section 3 we give a rather unconventional presentation of the stationary reductions of KdV. Our priviledged starting point is a picture of the KP hierarchy as a system of ordinary differential equations, called the *Central System* (CS) in [8, 15]. Starting from there, by means of a double reduction process we can describe quite explicitly the stationary reductions of KdV, and, in Section 4, we are able to give a Lax representation of these systems, with a Lax matrix depending polynomially on a parameter. This representation is used to show that the flows are bi-Hamiltonian. This is done in two steps. First, in Section 5 we recall the bi-Hamiltonian structure on matrix polynomials and show that the Hamiltonian vector fields (with respect to the Poisson pencil) admit a Lax formulation. This property is conserved after a suitable bi-Hamiltonian reduction process. Then, in Section 6, we identify the phase space of the stationary reductions of KdV with a reduced bi-Hamiltonian manifold, and we show that they are GZ systems. In Section 7 we state (referring to [16] for a more detailed discussion) a theorem ensuring that, under some additional assumptions, the GZ systems are separable in coordinates that are naturally associated with the bi-Hamiltonian structure. Finally, in Section 8 we show that this theorem can be applied to the stationary reductions of KdV, and that the variables of separation can be constructed algebraically.

Summing up, in this paper we present a somewhat self–contained approach to the study of stationary flows of KdV, and we use them as a laboratory to test ideas of the bi-Hamiltonian geometry, from the GZ theory to the separation of variables. In our opinion, such a set up provides a comprehensive formulation of results which, although for the most part already available in the literature, would perhaps acquire a deeper meaning under this perspective.

3 KdV Stationary Reductions

In this section we give a self-contained presentation of the stationary reductions of the KdV hierarchy, using the formalism developed in [8, 15] for the KP theory. Our starting point is the Central System (CS), a family of dynamical systems with $\mathbb{N} \times \mathbb{N}$ degrees of freedom. A first (stationary)

reduction gives rise to the CS_2 hierarchy, with $\mathbb{N}$ degrees of freedom. Then a further restriction leads to finite-dimensional systems that coincide with the stationary reductions of KdV.

We consider the space $\mathcal{H}$ of sequences $\{H^{(k)}\}_{k\geq 1}$ of Laurent series having the form $H^{(k)} = z^k + \sum_{l\geq 1} H^k_l z^{-l}$, where H^k_l are (complex) scalars that play the role of coordinates on $\mathcal{H}$. On such phase space $\mathcal{H}$ we define a family of vector fields as follows. We associate with a point $\{H^{(k)}\}_{k\geq 1}$ in $\mathcal{H}$ the linear span $H_+ = \langle H^{(0)}, H^{(1)}, H^{(2)}, \dots \rangle$, where $H^{(0)} = 1$. The *defining equation* for the j–th vector field X_j of the family, to be referred to as the *Central System* (CS), is the invariance relation

$$\left(\frac{\partial}{\partial t_j} + H^{(j)}\right) H_+ \subset H_+ \ . \tag{3.1}$$

This relation is equivalent to the (explicit) equations

$$\frac{\partial H^{(k)}}{\partial t_j} = -H^{(j)}H^{(k)} + H^{(j+k)} + \sum_{l=1}^{k} H^j_l H^{(k-l)} + \sum_{l=1}^{j} H^k_l H^{(j-l)}, \qquad k \geq 1. \tag{3.2}$$

Remark 3.1 From (3.2) it is evident that the exactness property

$$\frac{\partial}{\partial t_k} H^{(j)} = \frac{\partial}{\partial t_j} H^{(k)} \tag{3.3}$$

holds. Moreover, it can be shown [8] that the flows of the CS commute.

Remark 3.2 There is a very tight relation between the CS and the linear flows on the Sato Grassmannian [34, 35, 11]. This relation is discussed in [15], where the classical result of Sato on the linearization of the KP hierarchy is recovered from the point of view of the bi-Hamiltonian geometry.

Since the CS is a family of commuting vector fields, we can reduce it in many different ways. By means of a suitable combination of such reduction processes, the so-called fractional KdV hierarchies were obtained in [8]. Now we will show how the stationary reductions of KdV can be derived from the CS. The commutativity of the flows implies that the set $\mathcal{Z}_2$ of zeroes of the vector field X_2, defined by the quadratic equations

$$H^{(k+2)} - H^{(k)}H^{(2)} + \sum_{l=1}^{k} H^2_l H^{(k-l)} + H^k_1 H^{(1)} + H^k_2 = 0, \tag{3.4}$$

is an invariant submanifold for CS. Moreover, on $\mathcal{Z}_2$ we have

$$\frac{\partial H^{(2)}}{\partial t_j} = \frac{\partial H^{(j)}}{\partial t_2} = 0, \tag{3.5}$$

due to the exactness property (3.3). Therefore, the manifold $\mathcal{Z}_2$ is foliated by invariant submanifolds defined by the equation $H^{(2)} =$ constant. Among all these leaves, the submanifold $\mathcal{S}_2$ defined by the simple constraint

$$H^{(2)} = z^2 \tag{3.6}$$

is particularly relevant. At the points of $\mathcal{S}_2$ equation (3.4) takes the form

$$H^{(k+2)} = z^2 H^{(k)} - H_1^k H^{(1)} - H_2^k \ , \tag{3.7}$$

and allows us to recursively compute the Laurent coefficients of $H^{(k)}$, for $k > 2$, in terms of the coefficients of $h := H^{(1)}$. Hence, $\mathcal{S}_2$ is parametrized by the coefficients $\{h_l\}_{l\geq 1}$ of h. Equation (3.7) also shows that $z^2(H_+) \subset H_+$, so that on $\mathcal{S}_2$ the elements $\{z^{2j}, z^{2j}h\}_{j\geq 0}$ form a basis in H_+. Thus, we have that

$$H^{(k)} = p_k(z^2) + q_k(z^2)h \qquad \text{on } \mathcal{S}_2, \tag{3.8}$$

where p_k and q_k are polynomials. This can also be seen directly from equation (3.7). Moreover, there is only one Laurent series of the previous form satisfying the asymptotic condition $H^{(k)} = z^k + O(z^{-1})$ as $z \to \infty$.

Definition 3.3 (see [8]). *The restriction of CS to the invariant submanifold $\mathcal{S}_2$ is called the CS_2 hierarchy.*

The restricted vector fields are given by

$$\frac{\partial h}{\partial t_j} = -hH^{(j)} + H^{(j+1)} + \sum_{l=1}^{j} h_l H^{(j-l)} + H_1^j, \qquad j \geq 1, \tag{3.9}$$

where the $H^{(j)}$ must be written in terms of h according to (3.8). If we denote with H_- the span of the negative powers of z, and with π_- the projection on H_- according to the decomposition $H_+ \oplus H_-$, then the equations (3.9) can be written in the more compact form

$$\frac{\partial h}{\partial t_j} = -\pi_- \left(q_j h^2\right) \ . \tag{3.10}$$

Notice that $H^{(2k)} = z^{2k}$, so that (3.3) implies that the even flows of CS_2 are trivial.

The finite–dimensional systems that are the main subject of this paper are those obtained by restricting the CS_2 flows to the manifold of zeroes of the $(2g+1)$-st vector field of CS_2. We will call such systems the KdV_{2g+1} systems, since they are (equivalent to) the stationary reductions of the KdV hierarchy, as we are going to show at the end of this section. The constraint which defines the phase space $\mathcal{M}_{2g+1}$ of the KdV_{2g+1} system is

$$\frac{\partial h}{\partial t_{2g+1}} = -\pi_- \left(q_{2g+1} h^2\right) = 0 \ . \tag{3.11}$$

A direct inspection shows that this constraint gives all the coefficients of h in terms of the first $(2g+1)$, i.e., $h_1, \ldots, h_{2g+1}$. In other words, the dimension of the phase space of the KdV_{2g+1} system equals $2g+1$. The equations are given by the first $2g+1$ components of (3.10), after substituting the constraints (3.11). In the case of KdV_5 there are two independent vector fields:

$$\begin{aligned}
\frac{\partial h_1}{\partial t_1} &= -2h_2 & \frac{\partial h_1}{\partial t_3} &= -2h_4 + 2h_1h_2 \\
\frac{\partial h_2}{\partial t_1} &= -2h_3 - {h_1}^2 & \frac{\partial h_2}{\partial t_3} &= -2h_5 + {h_2}^2 + {h_1}^3 \\
\frac{\partial h_3}{\partial t_1} &= -2h_1h_2 - 2h_4 & \frac{\partial h_3}{\partial t_3} &= -2h_1h_4 + 4{h_1}^2h_2 - 2h_3h_2 \\
\frac{\partial h_4}{\partial t_1} &= -2h_5 - {h_2}^2 - 2h_1h_3 & \frac{\partial h_4}{\partial t_3} &= -2{h_3}^2 - 2h_2h_4 + 2h_1{h_2}^2 + {h_1}^4 + {h_1}^2h_3 \\
\frac{\partial h_5}{\partial t_1} &= -4h_3h_2 + 2{h_1}^2h_2 - 4h_1h_4 & \frac{\partial h_5}{\partial t_3} &= 2{h_1}^2h_4 - 4h_3h_4 + 2{h_1}^3h_2
\end{aligned} \tag{3.12}$$

These are, up to the coordinate change (3.14), the equations (2.4) and (2.5).

We remark that along the flows of KdV_{2g+1} the relations (3.3) take the form

$$\frac{\partial H^{(2g+1)}}{\partial t_j} = 0 \ , \tag{3.13}$$

showing that all the coefficients of $H^{(2g+1)}$ are integrals of motion. Therefore our presentation of the KdV stationary reductions carries directly the conserved quantities of the flows. Moreover, in the next section we will show that the Lax representation also arises in a natural way. We end this section with the following:

Remark 3.4 The usual KdV hierarchy in $1+1$ dimensions is described in [8] as a projection of CS_2 along the integral curves of the first vector field of the hierarchy,

$$\frac{\partial h}{\partial t_1} = -h^2 + z^2 + 2h_1 \ .$$

Indeed, if we put $x = t_1$ and $u = 2h_1$, then the previous equation takes the form $h_x + h^2 = u + z^2$ and allows us to write the h_j as polynomials in u and its x-derivatives:

$$\begin{aligned}
h_1 &= \tfrac{1}{2}u \\
h_2 &= -\tfrac{1}{4}u_x \\
h_3 &= \tfrac{1}{8}(u_{xx} - u^2) \\
h_4 &= -\tfrac{1}{16}(u_{xxx} - 4uu_x) \\
h_5 &= \tfrac{1}{32}(u_{xxxx} - 6uu_{xx} - 5u_x^2 + 2u^3) \\
&\vdots
\end{aligned} \tag{3.14}$$

Thus equations (3.9) become partial differential equations for the variable u, and are the KdV hierarchy. But we can also use the system (3.14) to recover CS_2 from the KdV hierarchy, so that we can pass back and forth from one hierarchy to the other. This shows that the KdV_{2g+1} systems that we have introduced coincide with the usual stationary reductions of KdV. The first $(2g+1)$ equations of the system (3.14) represent the change between our coordinates $(h_1, \ldots, h_{2g+1})$ and the ones usually considered in the literature, namely $(u, u_x, \ldots, u^{(2g+1)})$.

4 The Lax Representation

In this section we show that there is a quite natural (Zakharov-Shabat) zero-curvature representation for the CS_2 system, entailing a Lax representation for the KdV_{2g+1} hierarchy.

We know from the previous section that in the CS_2 theory every element in H_+ can be written as a linear combination of 1 and h with coefficients that are polynomials in $\lambda := z^2$. Then to each point of the manifold $\mathcal{S}_2$ (that is, to each series $h = z + \sum_{l\geq 1} h_l z^{-l}$) we can associate a family of 2×2 matrices $\mathsf{V}^{(j)}(\lambda)$ depending *polynomially* on λ, defined by the relation

$$\left(\frac{\partial}{\partial t_j} + H^{(j)}\right)\begin{bmatrix} 1 \\ h \end{bmatrix} = \mathsf{V}^{(j)}\begin{bmatrix} 1 \\ h \end{bmatrix}. \tag{4.1}$$

Since the even flows are trivial, we will be interested only in the matrices of odd index. The first three of them are given by

$$\mathsf{V}^{(1)} = \begin{bmatrix} 0 & 1 \\ \lambda + 2\,h_1 & 0 \end{bmatrix} \qquad \mathsf{V}^{(3)} = \begin{bmatrix} -h_2 & \lambda - h_1 \\ \lambda^2 + h_1\lambda + 2h_3 - {h_1}^2 & h_2 \end{bmatrix}$$

$$\mathsf{V}^{(5)} = \begin{bmatrix} -h_2\lambda - h_4 + h_1h_2 & \lambda^2 - h_1\lambda - h_3 + {h_1}^2 \\ \begin{array}{c}\lambda^3 + h_1\lambda^2 + h_3\lambda + \\ 2h_5 - 2h_1h_3 - {h_2}^2 + {h_1}^3\end{array} & h_2\lambda - h_1h_2 + h_4 \end{bmatrix}.$$

The commutativity of the flows and the "abelian" zero-curvature relation (3.3) imply that

$$\left(\frac{\partial}{\partial t_j}\mathsf{V}^{(k)} - \frac{\partial}{\partial t_k}\mathsf{V}^{(j)} + \left[\mathsf{V}^{(k)}, \mathsf{V}^{(j)}\right]\right)\begin{bmatrix} 1 \\ h \end{bmatrix} = 0. \tag{4.2}$$

Since the entries of the matrix appearing in the previous equation are polynomials in λ, and the elements $\{\lambda^j, \lambda^j h\}_{j\geq 0}$ are linearly independent in H_+, it follows that the zero–curvature relations

$$\frac{\partial}{\partial t_j}\mathsf{V}^{(k)} - \frac{\partial}{\partial t_k}\mathsf{V}^{(j)} + \left[\mathsf{V}^{(k)}, \mathsf{V}^{(j)}\right] = 0 \tag{4.3}$$

hold.

If we restrict to the set $\mathcal{M}_{2g+1}$ of the stationary points of the $(2g+1)$–st vector field of CS_2, the zero–curvature representation naturally gives rise to Lax equations for the matrix $\mathsf{V}^{(2g+1)}$,

$$\frac{\partial}{\partial t_k}\mathsf{V}^{(2g+1)} = \left[\mathsf{V}^{(k)}, \mathsf{V}^{(2g+1)}\right]. \tag{4.4}$$

The following proposition will be useful in Section 6, and shows that these Lax equations faithfully represent the KdV_{2g+1} system.

Proposition 4.1 *The matrices $\mathsf{V}^{(2k+1)}$ of the CS_2 hierarchy have the following properties:*

1. *The matrix $\mathsf{V}^{(2k+1)}$ depends only on $(h_1, \ldots, h_{2k+1})$, and the map $(h_1, \ldots, h_{2k+1}) \mapsto \mathsf{V}^{(2k+1)}$ is injective;*
2. *The trace of $\mathsf{V}^{(2k+1)}$ is zero;*
3. *For $i \leq k$ one has*

$$\mathsf{V}^{(2i+1)} = (\lambda^{i-k}\mathsf{V}^{(2k+1)})_+ - \alpha_{ik}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{4.5}$$

where $(\cdot)_+$ denotes the projection on the nonnegative powers of λ, and α_{ik} is the entry $(1,2)$ of the coefficient of λ^{k-i-1} in $\mathsf{V}^{(2k+1)}$.

Proof. First of all we observe that, almost by definition,

$$\mathsf{V}^{(2k+1)} = \begin{bmatrix} p_{2k+1} & q_{2k+1} \\ \begin{array}{c}\lambda^{k+1} + \sum_{l=0}^{k} h_{2l+1}\lambda^{k-l} + \\ \sum_{l=1}^{k} h_{2l}p_{2k-2l+1} + H_1^{2k+1}\end{array} & \sum_{l=1}^{k} h_{2l}q_{2k-2l+1} \end{bmatrix}. \tag{4.6}$$

Then, we notice that equation (3.7) implies the recursion formulas

$$p_{2k+1} = \lambda p_{2k_1} - H_2^{2k-1} , \qquad q_{2k+1} = \lambda q_{2k_1} - H_1^{2k-1} . \tag{4.7}$$

Thus, by induction, we obtain

$$p_{2k+1} = -\sum_{l=1}^{k} H_2^{2l-1}\lambda^{k-l} , \qquad q_{2k+1} = \lambda^k - \sum_{l=1}^{k} H_1^{2l-1}\lambda^{k-l} . \tag{4.8}$$

In order to express the coefficients H_2^{2l-1} and H_1^{2l-1} in terms of the h_l, we use the identity

$$H_l^{2k-1} = H^1_{2k+l-2} - \sum_{i=1}^{k+l-3} H^1_{l+2i-2}H_1^{2k-2i-1},$$

which can be proved by induction on k using again (3.7). In particular, we have

$$\begin{aligned} H_1^{2k-1} &= H^1_{2k-1} - \sum_{i=1}^{k-2} H^1_{2i-1}H_1^{2k-2i-1} && (4.9)\\ H_2^{2k-1} &= H^1_{2k} - \sum_{i=1}^{k-1} H^1_{2i}H_1^{2k-2i-1} . && (4.10)\end{aligned}$$

This allows us to control the appearance of the $h_i = H_i^1$ in p_{2l+1} and q_{2l+1}, and leads to the proof of the first assertion.

The second statement tantamounts to $p_{2k+1} + \sum_{l=1}^{k} h_{2l}q_{2k-2l+1} = 0$, and is easily proved by inserting (4.8) and using (4.10).

As far as the last assertion is concerned, we use the following consequences of (4.8):

$$(\lambda^{i-k}p_{2k+1})_+ = p_{2i+1}, \qquad (\lambda^{i-k}q_{2k+1})_+ = q_{2i+1} . \tag{4.11}$$

This gives, using (4.6),

$$(\lambda^{i-k}\mathsf{V}^{(2k+1)})_+ = \mathsf{V}^{(2i+1)} - H_1^{2i+1}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} . \tag{4.12}$$

Since $-H_1^{2i+1}$ is the coefficient of λ^{k-i-1} in q_{2k+1}, equation (4.6) shows that we are done.

$\square$

So we have seen that the double reduction process of the Central System outlined in Section 3 provides us with a natural Lax representation of the (commuting) vector fields of the KdV_{2g+1} system. Actually, as it was explained in [15], the Central System can be seen as an outgrowth of the bi-Hamiltonian properties of the KdV hierarchy. It is thus natural to look for a bi-Hamiltonian structure of the KdV_{2g+1} system. Unfortunately, we are not in a position to derive such a property from the Central System itself, but rather we have to rely on the Lax representation discussed so far. Namely, in the next two sections we will establish the bi-Hamiltonian nature of KdV_{2g+1}, showing that it comes from the general theory of bi-Hamiltonian systems defined on matrices depending polynomially on a parameter.

5 Lax Equations and bi-Hamiltonian Systems

In the previous section we have associated with every point of the phase space $\mathcal{M}_{2g+1}$ of KdV_{2g+1} a Lax matrix $\mathsf{V}^{(2g+1)}$, and we have seen that this matrix gives a Lax representation of the flows. To give these flows a bi-Hamiltonian formulation, we will address in this section a general problem, concerning the relation between Lax matrices and bi-Hamiltonian structures. We will describe a class of bi-Hamiltonian manifolds whose (bi-)Hamiltonian flows have a Lax formulation, and show that this formulation survives a reduction process of Marsden-Ratiu type. Since $\mathsf{V}^{(2g+1)}$ depends polynomially on λ, it is quite natural to consider the multi-Hamiltonian structures defined on $\mathfrak{g}$–valued polynomials (see [33, 24]), where $\mathfrak{g}$ is a Lie algebra of matrices such that the trace of the product is nondegenerate. More precisely, for a fixed matrix $A \in \mathfrak{g}$, let us consider the space

$$\mathcal{M}_A := \{X(\lambda) = \lambda^{n+1}A + \sum_{i=0}^{n} \lambda^i X_i \mid X_i \in \mathfrak{g}\} \ , \tag{5.1}$$

which is clearly in a 1-1 correspondence with the space $\bigoplus_{i=0}^n \mathfrak{g}$ of $(n+1)$–tuples of matrices in $\mathfrak{g}$. The tangent and the cotangent space at a point of $\mathcal{M}_A$ can also be identified with $\bigoplus_{i=0}^n \mathfrak{g}$, using the pairing

$$\langle (V_0,\ldots,V_n),(W_0,\ldots,W_n)\rangle = \sum_{i=0}^{n} \mathrm{Tr}(V_i W_i) \ . \tag{5.2}$$

If F is a function on $\mathcal{M}_A$, we will denote its differential by

$$dF = \left(\frac{\partial F}{\partial X_0},\ldots,\frac{\partial F}{\partial X_n}\right) \ .$$

It is known that on $\mathcal{M}_A$ there is an $(n+2)$–dimensional web of (compatible) Poisson brackets, and that this web is associated with a family of classical R–matrices. Nevertheless, it turns out that in our case the relevant Poisson pair is given by the first two brackets of the above mentioned family. The first Poisson tensor, as a map from the cotangent to the tangent space, is given by

$$P_0 : \begin{pmatrix} W_0 \\ W_1 \\ \vdots \\ \vdots \\ W_{n-1} \end{pmatrix} \mapsto \begin{pmatrix} \dot X_0 \\ \dot X_1 \\ \vdots \\ \vdots \\ \dot X_{n-1} \end{pmatrix} = \begin{pmatrix} [X_1,\cdot] & [X_2,\cdot] & \cdots & \cdots & [A,\cdot] \\ [X_2,\cdot] & \cdots & \cdots & [A,\cdot] & 0 \\ \vdots & \cdots & \cdot & & \\ \vdots & \cdot & & & \\ [A,\cdot] & 0 & \cdots & & 0 \end{pmatrix} \begin{pmatrix} W_0 \\ W_1 \\ \vdots \\ \vdots \\ W_{n-1} \end{pmatrix} \ , \tag{5.3}$$

while the second one is

$$P_1 : \begin{pmatrix} W_0 \\ W_1 \\ \vdots \\ \vdots \\ W_{n-1} \end{pmatrix} \mapsto \begin{pmatrix} \dot X_0 \\ \dot X_1 \\ \vdots \\ \vdots \\ \dot X_{n-1} \end{pmatrix} = \begin{pmatrix} -[X_0,\cdot] & 0 & \cdots & \cdots & 0 \\ 0 & [X_2,\cdot] & [X_3,\cdot] & \cdots & [A,\cdot] \\ 0 & [X_3,\cdot] & \cdots & \cdot & \\ \vdots & \vdots & \cdot & & \\ 0 & [A,\cdot] & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} W_0 \\ W_1 \\ \vdots \\ \vdots \\ W_{n-1} \end{pmatrix} \ . \tag{5.4}$$

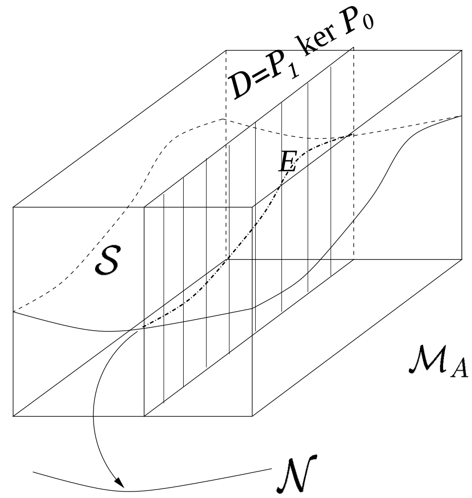

Figure 1: The bi-Hamiltonian reduction

The associated Poisson brackets are given by $\{F,G\}_i = \langle dF, P_i dG\rangle$, where $i = 1,2$, and F, G are functions on $\mathcal{M}_A$. The Poisson tensors (5.3) and (5.4) satisfy the remarkable property that every linear combination of them is still a Poisson tensor. For this reason one says that they are *compatible*, and that $\mathcal{M}_A$ is a bi-Hamiltonian manifold.

Let us consider now the Poisson pencil $P_\lambda := P_1 - \lambda P_0$. It is important to notice that its Hamiltonian vector fields admit a Lax representation, as shown in the following:

Proposition 5.1 *Let F be a function on $\mathcal{M}_A$, and $\frac{\partial X(\lambda)}{\partial t_\lambda} = P_\lambda dF$ the Hamiltonian vector field associated by P_λ to F. Then,*

$$\frac{\partial X(\lambda)}{\partial t_\lambda} = \left[\frac{\partial F}{\partial X_0}, X(\lambda)\right] . \tag{5.5}$$

Proof. Use the expressions (5.3) and (5.4) to compute the vector field $P_\lambda dF$, then identify the parameter λ appearing in the Poisson pencil with the λ in $X(\lambda)$.

□

At this point we want to enlarge the class of bi-Hamiltonian manifolds giving rise to systems admitting a Lax representation, having also in mind the case of the stationary reductions of KdV. To this aim, it is important to recall a reduction theorem [9] allowing us to "move" the bi-Hamiltonian structure from a given ("big") manifold to a smaller one. This result is a particular case of a theorem by Marsden and Ratiu for Poisson manifolds [26], and can be applied to a general bi-Hamiltonian manifold. Here, for the sake of simplicity, we will describe it only in the case at hand. The central point is that a Lax representation can be found also for the vector fields that are Hamiltonian with respect to the reduced Poisson pencil.

The first step of the reduction process is to fix a symplectic leaf $\mathcal{S}$ of P_0. Then we introduce the distribution $D = P_1(\operatorname{Ker} P_0)$, which is integrable thanks to the compatibility between P_0 and P_1. From the explicit form (5.3)–(5.4) of the Poisson tensors, it is easy to see that the vector fields in D have the Lax form $\dot{X}(\lambda) = [W_0, X(\lambda)]$, for a suitable W_0. Let us denote by E the intersection of D with $T\mathcal{S}$. The statement of the bi-Hamiltonian reduction theorem is that the quotient manifold

$\mathcal{N} = \mathcal{S}/E$ inherits from $\mathcal{M}_A$ a bi-Hamiltonian structure. In order to compute the reduced Poisson bracket $\{f, g\}_\lambda$ between two functions f, g on $\mathcal{N}$, we consider them as functions on $\mathcal{S}$, invariant along the leaves of E. Then, we choose functions F and G on $\mathcal{M}_A$ which extend f and g, and annihilate the distribution D. Their Poisson bracket $\{F, G\}_\lambda$ is still invariant along D, and therefore defines a function on $\mathcal{N}$, which is independent of the choice of the prolongations F and G.

Let us consider now a given Hamiltonian vector field X_f (with respect to the Poisson pencil) on $\mathcal{N}$, with Hamiltonian f. If F is a prolongation of f, the vector field $X_F := P_\lambda dF$ is easily seen to be tangent to $\mathcal{S}$ and to project onto X_f. We are going to show that X_f inherits a Lax representation from the one of X_F. To this aim, we suppose that there exists a submanifold $\mathcal{Q}$ of $\mathcal{S}$, which is transversal to the distribution E. In other words, $\mathcal{Q}$ is the image of a section of the bundle $\pi : \mathcal{S} \to \mathcal{N}$. Then $\mathcal{Q}$ is diffeomorphic to $\mathcal{N}$, and inherits a bi-Hamiltonian structure. The representative of X_f on $\mathcal{Q}$ is simply found by decomposing the restriction of X_F to $\mathcal{S}$ according to the splitting $T\mathcal{S} = T\mathcal{Q} \oplus E$. Since we have seen that E is spanned by vector fields having a Lax form, we have proved the following:

Proposition 5.2 *Let $\mathcal{Q} \subset \mathcal{S}$ be transversal to E. Then, the vector fields on $\mathcal{N}$ which are Hamiltonian with respect to the Poisson pencil admit a Lax representation on $\mathcal{Q}$.*

Therefore the bi-Hamiltonian reduction implies, in this case, a reduction of the Lax formulation.

6 The bi-Hamiltonian Structure of the KdV Stationary Reductions

The aim of this section is to show that the KdV_{2g+1} systems introduced in Section 3 admit a bi-Hamiltonian formulation. To do this, we are going to exploit the Lax representation found in Section 4 and the results of the preceding section.

The form of the Lax matrix $\mathsf{V}^{(2g+1)}$ found in Section 3 suggests to choose $\mathfrak{g} = \mathfrak{sl}(2)$, $n = g$, and

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} .$$

Therefore, the dimension of $\mathcal{M}_A$ is $3(g+1)$. The Lax matrix $\mathsf{V}^{(2g+1)}$ defines an embedding of the KdV_{2g+1} phase space into $\mathcal{M}_A$. At this point two natural questions arise:

1. Does this submanifold inherit from $\mathcal{M}_A$ the bi-Hamiltonian structure?
2. If so, are the vector fields of KdV_{2g+1} bi-Hamiltonian with respect to this structure?

We will see that the answer to both questions is yes.

In order to answer the first question we need a careful description of the symplectic leaves of P_0, as given by

Lemma 6.1 *The symplectic leaves of P_0 have dimension $2(g+1)$. Moreover, let $H(\lambda) : \mathcal{M}_A \to \mathbb{C}$ be defined as $H(X(\lambda)) := \frac{1}{2}\mathrm{Tr} X(\lambda)^2$ and let the H_i be the coefficient of λ^i in $H(\lambda)$. Then, the functions $H_{2g+1}, \ldots, H_{g+1}$ are functionally independent Casimirs of P_0. Consequently, the symplectic leaves of P_0 are the level surfaces of the previous Casimirs.*

Proof. From (5.3) the kernel of P_0 is easily seen to be given by the covectors $[W_0, \ldots, W_g]^T$ such that $W_i = \alpha_i A + \sum_{l=1}^{i} \alpha_{i-l} X_{g+1-l}$, where the α_i, $i = 0, \ldots, g$, are arbitrary. This shows that

$\dim(\mathrm{Ker}P_0) = n$, so that the dimension of the symplectic leaves is $2(g+1)$. In order to check that $H_{2g+1}, \ldots, H_{g+1}$ are Casimirs, it is sufficient to verify that the differential of H_i is the 1-form $[X_i, X_{i-1}, \ldots, X_{i-g}]^T$, where $X_k := 0$ if $k < 0$.

□

One can easily show that the symplectic leaf defined by $H_i = c_i$, with $g+1 \le i \le 2g+1$, can be parametrized as

$$X(\lambda) = \lambda^{g+1} A + \sum_{j=0}^{g} \lambda^j \begin{bmatrix} p_j & r_j \\ q_j & -p_j \end{bmatrix} , \tag{6.1}$$

where p_j and q_j are free parameters, and r_j is a function of $(p_{j+1}, q_{j+1}, \ldots, p_g, q_g)$ and the values $(c_{g+j+1}, \ldots, c_{2g+1})$ of the Casimirs.

As far as the distribution $D = P_1(\mathrm{Ker}P_0)$ is concerned, in this case it is tangent to the symplectic leaves of P_0. Indeed, from the explicit form (5.3)–(5.4) of the Poisson tensors it is easy to see that D is the 1-dimensional distribution spanned by the vector field

$$\dot{X}(\lambda) = [A, X(\lambda)] . \tag{6.2}$$

This also shows that the integral leaves of D are simply the orbit of the action given by simultaneous conjugation of the isotropy subgroup of A, but we will never use this fact.

Now we are ready to endow the phase space $\mathcal{M}_{2g+1}$ of KdV_{2g+1} with the structure of a bi-Hamiltonian manifold. This follows from the fact that the map assigning to each point of $\mathcal{M}_{2g+1}$ the corresponding Lax matrix $\mathsf{V}^{(2g+1)}$ defines a submanifold of a suitable symplectic leaf of P_0, which is transversal to the distribution E. This is shown in the following:

Proposition 6.2 *Let us take the symplectic leaf $\widetilde{\mathcal{S}}$ defined by $H_{2g+1} = 1$ and $H_i = 0$ for $g+1 \le i \le 2g$. Then, $\mathsf{V}^{(2g+1)}(h_1, \ldots, h_{2g+1}) \in \widetilde{\mathcal{S}}$ for all $(h_1, \ldots, h_{2g+1}) \in \mathcal{M}_{2g+1}$. Moreover, the image $\widetilde{\mathcal{Q}}$ of the previous map is transversal to E.*

Proof. By the definition of $\widetilde{\mathcal{S}}$, we have to show that $\frac{1}{2}\mathrm{Tr}\left(\mathsf{V}^{(2g+1)}\right)^2 = \lambda^{2g+2} + \sum_{i=0}^{g} H_i \lambda^i$. To this aim, we observe that equation (4.1) and the stationarity of t_{2g+1} imply that $H^{(2g+1)}(z)$ is an eigenvalue of $\mathsf{V}^{(2g+1)}$. The other eigenvalue is given by $H^{(2g+1)}(-z)$, because $\mathsf{V}^{(2g+1)}$ depends only on $\lambda = z^2$. Therefore, $\mathrm{Tr}\left(\mathsf{V}^{(2g+1)}\right)^2 = (H^{(2g+1)}(z))^2 + (H^{(2g+1)}(-z))^2$ has the desired form. Finally, referring to the parametrization (6.1), one can easily prove that the submanifold $p_g = 0$ is transversal to the distribution E.

□

Hence, the KdV_{2g+1} phase space inherits from $\widetilde{\mathcal{Q}}$ (and from the quotient space $\widetilde{\mathcal{N}} = \widetilde{\mathcal{S}}/E$) a bi-Hamiltonian structure. To compute this structure, it is convenient to use the formalism discussed in [10], whose aim is to avoid dealing with the explicit form of the projection $\pi : \widetilde{\mathcal{S}} \to \widetilde{\mathcal{N}}$.

Now we want to show that the KdV_{2g+1} flows are indeed bi-Hamiltonian with respect to the above Poisson pencil. The Lax representation (4.4) and the form (4.5) of the Lax pair suggest to consider the vector fields on $\mathcal{M}_A$ given by

$$\frac{\partial X(\lambda)}{\partial t_i} = \left[\left(\lambda^{i-g} X(\lambda)\right)_+ , X(\lambda) \right] , \qquad i = -1, 0, \ldots, g-1. \tag{6.3}$$

They are Hamiltonian with respect to the Poisson pencil on $\mathcal{M}_A$, with Hamiltonian function given by $X(\lambda) \mapsto \left(\lambda^{i-g} H(X(\lambda))\right)_+$, where $H(X(\lambda)) = \frac{1}{2}\mathrm{Tr} X(\lambda)^2$. Furthermore, we can state

Proposition 6.3 *On the bi-Hamiltonian manifold $\mathcal{M}_A$ the function $H(\lambda) = \sum_{i=0}^{2g+1} H_i\lambda^i$ is a Casimir of the Poisson pencil. The bi-Hamiltonian vector field $Y_{g-i} := P_0 dH_{i-1} = P_1 dH_i$ has the Lax representation (6.3).*

Proof. One can easily see that $dH(\lambda) = [X(\lambda), \lambda X(\lambda), \ldots, \lambda^g X(\lambda)]^T$. Thus from (5.3) and (5.4) it follows that $P_\lambda dH(\lambda) = 0$. The vector fields Y_i are Hamiltonian with respect to P_λ, since $Y_{g-i} = P_1 dH_i = P_\lambda d(\lambda^{-i} H(\lambda))_+$. Thus the Lax representation (6.3) is a consequence of Proposition 5.1.

□

Remark 6.4 The previous proposition is a particular case of a general result [33], stating that Ad-invariant polynomial functions on a Lie algebra $\mathfrak{g}$ give rise to Casimir of the Poisson pencil on $\mathcal{M}_A$.

In order to obtain the KdV_{2g+1} system from the bi-Hamiltonian vector fields (6.3), we remark that, from general results of the bi-Hamiltonian theory:

1. The functions H_i are invariant along the distribution E, and therefore can be projected on the quotient $\widetilde{\mathcal{N}}$.
2. The vector fields (6.3) are tangent to $\widetilde{\mathcal{S}}$ and project onto $\widetilde{\mathcal{N}}$.
3. Their projections are the bi-Hamiltonian vector fields associated with the projected functions.

We observe from (6.3) that the vector field $Y_{-1} = P_0 dH_g$ is tangent to the distribution E. This means that the function H_g, on the quotient $\widetilde{\mathcal{N}}$, is a Casimir of the reduction of P_0. Hence, the polynomial $H_0 + H_1\lambda + \cdots + H_g\lambda^g$ is a Casimir of the reduced Poisson pencil. The other vector fields in (6.3) project on the stationary reductions of KdV, as shown in

Proposition 6.5 *The projections on $\widetilde{\mathcal{N}}$ of the vector fields (6.3), for $i = 0, \ldots, g-1$, coincide with the KdV_{2g+1} systems.*

Proof. The right place to compare the two hierarchies of vector fields is the transversal submanifold $\widetilde{\mathcal{Q}}$ defined as the image of the map $\mathsf{V}^{(2g+1)} : \mathcal{M}_{2g+1} \to \widetilde{\mathcal{S}}$. Hence, we must project the Lax equations (6.3) on $T\widetilde{\mathcal{Q}}$ along E. This leads to

$$\left[\left(\lambda^{i-g} X(\lambda)\right)_+, X(\lambda)\right] - \alpha\,[A, X(\lambda)]\ , \tag{6.4}$$

where α is fixed by the condition that this vector be tangent to $\widetilde{\mathcal{Q}}$. This means that the entry $(1,1)$ of the coefficient of λ^g in (6.4) must be zero. If we write $\left[\left(\lambda^{i-g} X(\lambda)\right)_+, X(\lambda)\right] = \left[X(\lambda), \left(\lambda^{i-g} X(\lambda)\right)_-\right]$, then we obtain that α is the entry $(1,2)$ of the λ^{g-i-1}–coefficient of $X(\lambda)$, so that equation (4.5) concludes the proof.

□

Therefore, we have shown that the KdV_{2g+1} flows are bi-Hamiltonian. Moreover, they are associated with a Casimir of the Poisson pencil, having a polynomial dependence on λ. Since this is a particular instance of a general theory developed by Gel'fand and Zakharevich [19], we will say that the vector fields of KdV_{2g+1} are GZ systems. The next section is devoted to the separability of such systems.

7 Separability of bi-Hamiltonian Systems

We have just seen that the stationary reductions of KdV are examples of GZ systems. In this section we show that the bi-Hamiltonian structure of such systems allows one to solve them by separation of variables. Under special circumstances, separability of GZ systems was proven in [5, 27]. We refer to [16] for complete proofs and a more detailed discussion.

Let $\mathcal{M}$ be a $(2n+1)$-dimensional manifold endowed with a pencil $P_\lambda = P_1 - \lambda P_0$ of Poisson tensors. We suppose that the rank of P_λ is generically $2n$, so that (locally) there exists a polynomial Casimir function $H(\lambda) = \sum_{i=0}^{n} H_i \lambda^i$ of P_λ (see [19]). The associated GZ systems $P_0 dH_i = P_1 dH_{i+1}$ are obviously tangent to the symplectic leaves of P_0, and give rise to Liouville integrable systems. Since H_n is a Casimir of P_0, such leaves are the level surface of H_n. Let us denote by ω the symplectic form given by the restriction of P_0 to a (fixed) symplectic leaf; if $X_f := P_0 df$, where f is any function on $\mathcal{M}$, then

$$\omega(X_f, X_g) = \{f, g\}_0 \ .$$

In order to exploit the existence of the other Poisson bracket, we make an additional assumption. We suppose that there exists a vector field Z on $\mathcal{M}$ such that

1. It is transversal to the symplectic leaves of P_0;

2. The functions invariant along Z form a *Poisson subalgebra* with respect to the bracket $\{\cdot,\cdot\}_\lambda$ associated with the Poisson pencil.

The second condition means that the bi-Hamiltonian structure can be projected on the quotient space of the integral leaves of Z. The first one tells us that such quotient can be identified with a symplectic leaf $\mathcal{S}_c$ of P_0, which is therefore a bi-Hamiltonian manifold. Moreover, we can define on $\mathcal{S}_c$ a *Nijenhuis tensor* N as

$$\omega(X_f, NX_g) = \{f, g\}_1 \ ,$$

where f and g are functions on $\mathcal{M}$ invariant along Z. Thus $\mathcal{S}_c$ is said to be a Poisson–Nijenhuis (PN) manifold (see [21] and references cited therein).

The vector field Z allows us to use the Poisson pencil to construct variables of separation for the GZ systems on $\mathcal{S}_c$. Hence in this case the Poisson pencil not only provides us with a commuting family of vector fields, but also gives coordinates for which the corresponding equations of motion can be solved by separation of variables. Indeed, one can show that the Nijenhuis tensor N has n functionally independent eigenvalues $(\lambda_1, \dots, \lambda_n)$. Then (see, e.g., [25]), there exist n complementary coordinates $(\mu_1, \dots, \mu_n)$ on $\mathcal{S}_c$ such that ω takes the canonical form $\omega = \sum_{i=1}^{n} d\lambda_i \wedge d\mu_i$ and the adjoint N^* of N takes the diagonal form

$$N^* d\lambda_j = \lambda_j d\lambda_j, \qquad N^* d\mu_j = \lambda_j d\mu_j \ .$$

Such coordinates are called *Darboux–Nijenhuis (DN) coordinates*. They are the separating coordinates for the GZ systems. Indeed, let us normalize Z in such a way that $Z(H_n) = 1$. Then the differentials of the restrictions $\widehat{H}_i$ of the Hamiltonians H_i to $\mathcal{S}_c$ generate a subspace which is

invariant with respect to N^*. More precisely, we have that

$$\begin{bmatrix} N^* d\widehat{H}_0 \\ \vdots \\ \vdots \\ \vdots \\ N^* d\widehat{H}_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & c_0 \\ 1 & 0 & \cdots & & c_1 \\ 0 & 1 & \cdot & & c_2 \\ \vdots & & \cdot & & \vdots \\ 0 & 0 & \cdots & 1 & c_{n-1} \end{bmatrix} \begin{bmatrix} d\widehat{H}_0 \\ \vdots \\ \vdots \\ \vdots \\ d\widehat{H}_{n-1} \end{bmatrix}, \tag{7.1}$$

where $c_i = -Z(H_i)$. This implies also that

$$\text{minimal polynomial of } N = \lambda^n - \sum_{i=0}^{n-1} c_i \lambda^i = Z(H(\lambda)) \ . \tag{7.2}$$

Moreover, one can check that the Frobenius matrix F defined by (7.1) satisfies the condition

$$N^* d\mathsf{F} = \mathsf{F}\, d\mathsf{F} \ , \tag{7.3}$$

where $d\mathsf{F}$ is the matrix whose entries are the differentials of the entries of F, and, on the left–hand side, N^* acts separately on each entry. Conditions (7.3) and (7.1) imply that the Hamilton-Jacobi equations for the $\widehat{H}_i$ are (collectively) separable in the DN coordinates. In fact, the (transpose of the) Vandermonde matrix constructed with the λ_j diagonalizes F, and applied to $[\widehat{H}_0, \cdots, \widehat{H}_{n-1}]^T$ gives a *Stäckel vector*, in the sense that its j–th component depends only on (λ_j, μ_j).

As far as the *explicit construction* of the DN coordinates is concerned, we have seen that the λ_j are the roots of the minimal polynomial $Z(H(\lambda))$ of N. On the contrary, the coordinates μ_j must be computed (in general) by a method involving quadratures. However, in the case at hand there is a recipe that is particularly useful in the applications. Let us consider the Hamiltonian vector field Y on $\mathcal{S}_c$, associated with $\frac{1}{2}\mathrm{Tr}N = \sum_{i=1}^n \lambda_i$ by the symplectic form ω. If the (restriction of the) Casimir $\widehat{H}(\lambda)$ satisfies the condition $Y^r(\widehat{H}(\lambda)) = 0$ for some r, then the coordinates

$$\mu_j = \frac{Y^{r-2}(\widehat{H}(\lambda_j))}{Y^{r-1}(\widehat{H}(\lambda_j))}$$

form with the λ_j a set of DN coordinates. Hence in this case the bi-Hamiltonian structure provides us with a method to algebraically construct the separation variables.

8 Separability of the Stationary Reductions

In this section we will show that the KdV$_{2g+1}$ system belongs to the class of separable GZ systems discussed above. It is convenient to show that the conditions on the vector fields Z and Y are fulfilled on the "big" bi-Hamiltonian manifold $\mathcal{M}_A$ and then to reduce everything.

Regarding the transversal vector field Z, we introduce on $\mathcal{M}_A$ the vector field $Z^{\mathcal{M}_A}$ defined as

$$\dot{X}_0 = A, \qquad \dot{X}_i = 0 \ \text{ for all } i = 1, \ldots, g. \tag{8.1}$$

It is tangent to the symplectic leaves of the Poisson tensor P_0 given by (5.3) (since it is easily seen to belong to its image) and it can be projected on the quotient space $\widetilde{\mathcal{N}} = \widetilde{\mathcal{S}}/E$ (since it commutes

with the generator (6.2) of the distribution E). Using again the form (5.3) and (5.4) of the Poisson tensors on $\mathcal{M}_A$, one can check that the functions invariant along $Z^{\mathcal{M}_A}$ form a Poisson subalgebra with respect to P_λ. This property is trivially conserved after the reduction on $\widetilde{\mathcal{N}}$. Finally, we have to show that the reduced vector field is transversal to the symplectic leaves of (the reduction of) P_0. This follows from the fact that, at the points where $\mathrm{Tr}(X_g A) = 1$,

$$L_{Z^{\mathcal{M}_A}}(\tfrac{1}{2}\mathrm{Tr}X(\lambda)^2) = \mathrm{Tr}\,(X(\lambda)A) = \lambda^g + \cdots\ ,$$

so that $L_{Z^{\mathcal{M}_A}} H_g = 1$. This also shows that the reduction of $Z^{\mathcal{M}_A}$ has the right normalization.

Thus, we have shown that on $\widetilde{\mathcal{N}}$, which is diffeomorphic to the phase space of KdV_{2g+1}, there exists a vector field that satisfies the hypotheses of the previous section. Therefore, the stationary reductions of KdV can be solved by separation of variables in the DN coordinates. We are left with the problem of finding explicitly these coordinates.

To this aim, we introduce the vector field $Y^{\mathcal{M}_A}$ defined on $\mathcal{M}_A$ as

$$\dot{X}_0 = [A, X_g], \qquad \dot{X}_i = 0 \ \text{ for all } i = 1, \dots, g. \tag{8.2}$$

It is also tangent to the symplectic leaves of the Poisson tensor P_0 and can be projected on the quotient space $\widetilde{\mathcal{N}}$. Moreover,

1. $Y^{\mathcal{M}_A}$ is (up to a sign) the Hamiltonian vector field associated by means of P_0 with the Lie derivative along $Z^{\mathcal{M}_A}$ of the coefficient H_{g-1} of $\frac{1}{2}\mathrm{Tr}X(\lambda)^2$;

2. We have that $L^2_{Y^{\mathcal{M}_A}}(H(\lambda)) = 2\,(\mathrm{Tr}(AX_g))^2$.

The first assertion can be checked after noticing that $Z^{\mathcal{M}_A}(H_{g-1}) = \mathrm{Tr}(X_{g-1}A)$, and that the differential of this function is $[0, \dots, 0, A, 0]^T$. The second assertion simply follows from the fact that $H(\lambda) = \frac{1}{2}\mathrm{Tr}X(\lambda)^2$.

The same properties hold also on $\widetilde{\mathcal{N}}$: If Z and Y are the reductions of $Z^{\mathcal{M}_A}$ and $Y^{\mathcal{M}_A}$, we have that Y is the Hamiltonian vector field associated with $-Z(H_{g-1})$ by the reduction of P_0. Furthermore, we have that $L^2_Y(H(\lambda)) = 2$, since $\mathrm{Tr}(AX_g) = 1$ at the points of $\widetilde{\mathcal{S}}$. Let us now restrict ourselves to a symplectic leaf of the reduction of P_0. Since $-Z(H_{g-1}) = c_{g-1} = \frac{1}{2}\mathrm{Tr}N$, the vector field Y can be used to construct the μ_j coordinates according to the recipe given at the end of the previous section. The conclusion is:

1. The λ_j are the roots of the polynomial $Z(H(\lambda)) = \mathrm{Tr}(AX(\lambda)) = \mathrm{Tr}(A\mathsf{V}^{(2g+1)})$, that is, the entry $(1, 2)$ of the matrix $\mathsf{V}^{(2g+1)}$;

2. The μ_j are given by $\mu_j = f(\lambda_j)$, where

$$f(\lambda) = \frac{Y(H(\lambda))}{Y^2(H(\lambda))} = \frac{1}{2}\mathrm{Tr}\,(X(\lambda)[A, X_g]) = \text{entry } (2,2) \text{ of } \mathsf{V}^{(2g+1)}.$$

We remark that our general theory of separability gives, in this particular case, the same construction of the variables of separation holding for systems admitting Lax with parameter formulation (see, e.g., [1, 36]). In fact, writing the Lax matrix (4.6) as

$$\mathsf{V}^{(2g+1)} = \begin{bmatrix} V_g(\lambda) & U_g(\lambda) \\ W_g(\lambda) & -V_g(\lambda) \end{bmatrix},$$

the equation of the associated spectral curve C is

$$\mu^2 = U_g(\lambda)W_g(\lambda) + V_g(\lambda)^2 \ .$$

Since $U_g(\lambda_j) = 0$ and $\mu_j = -V_g(\lambda_j)$, we see that g points (λ_j, μ_j) lie on C.

We close this section with a description, from our point of view, of the example of KdV_5 we started with in Section 2. We consider the bi-Hamiltonian manifold

$$\mathcal{M}_A = \left\{ A\lambda^3 + X_2\lambda^2 + X_1\lambda + X_0 \mid X_i = \begin{bmatrix} p_i & r_i \\ q_i & -p_i \end{bmatrix} \right\} ,$$

whose Poisson tensors are given by (5.3) and (5.4). The reduction process described in Section 5 allows us to pass to the transversal submanifold

$$A\lambda^3 + \begin{bmatrix} 0 & 1 \\ q_2 & 0 \end{bmatrix}\lambda^2 + \begin{bmatrix} p_1 & -q_2 \\ q_1 & -p_1 \end{bmatrix}\lambda + \begin{bmatrix} p_0 & {q_2}^2 - q_1 \\ q_0 & -p_0 \end{bmatrix} ,$$

which is diffeomorphic to the phase space $\mathcal{M}_5$ of KdV_5. The correspondence is given through the Lax matrix $\mathsf{V}^{(5)}$ displayed in Section 4. The resulting change of variables is explicitly given by

$$\begin{gathered} h_1 = q_2 \ , \qquad h_2 = -p_1 \ , \qquad h_3 = q_1 \ , \qquad h_4 = -p_0 - p_1q_2 \ , \\ h_5 = q_1q_2 + \tfrac{1}{2}{p_1}^2 - \tfrac{1}{2}{q_2}^3 + \tfrac{1}{2}q_0 \ . \end{gathered}$$

The Poisson pencil on $\mathcal{M}_5$ turns out to be

$$P_\lambda = \begin{bmatrix} 0 & -1 & 0 & -h_1+\lambda & -h_2 \\ 1 & 0 & 2h_1-\lambda & h_2 & h_3 + \frac{1}{2}{h_1}^2 - 2h_1\lambda \\ 0 & -2h_1+\lambda & 0 & -h_3 - {h_1}^2 + 2h_1\lambda & -h_4 - h_1h_2 \\ h_1-\lambda & -h_2 & * & 0 & \begin{matrix} -h_5 + 3h_1h_3 - \frac{1}{2}{h_2}^2 - {h_1}^3 - \\ (2h_3 + {h_1}^2)\lambda \end{matrix} \\ * & * & * & * & 0 \end{bmatrix} \tag{8.3}$$

Its Casimir $H(\lambda) = H_0 + H_1\lambda + H_2\lambda^2$ can be computed with the trace of the square of the Lax matrix:

$$\begin{aligned} H_0 &= h_3{h_2}^2 - 2h_3h_5 + {h_1}^5 + 2h_1{h_3}^2 - 2h_1h_2h_4 - 3{h_1}^3h_3 + 2{h_1}^2h_5 + {h_4}^2 \\ H_1 &= 2h_2h_4 - 2h_1h_5 + 3{h_1}^2h_3 - h_1{h_2}^2 - {h_3}^2 - {h_1}^4 \\ H_2 &= 2{h_1}^3 - 4h_1h_3 + 2h_5 \end{aligned}$$

The two vector fields of the KdV_5 hierarchy are given by (3.12). The symplectic leaves of P_0 are the level surfaces of H_2. The vector field $Z^{\mathcal{M}_A}$ is $\partial/\partial q_0$, while its projection Z on $\mathcal{M}_5$ is $(1/2)\partial/\partial h_5$. On the symplectic leaf $\mathcal{S}_c$ defined by $H_2 = c$ we can use (h_1, h_2, h_3, h_4) as global coordinates, and the corresponding Poisson pencil is simply obtained by deleting the last row and the last column in (8.3). The minimal polynomial of the Nijenhuis tensor on $\mathcal{S}_c$ is

$$Z(H(\lambda)) = \lambda^2 - h_1\lambda + h_1^2 - h_3 \ ,$$

and λ_1, λ_2 are its roots. To find μ_1 and μ_2 we have to use

$$Y^{\mathcal{M}_A} = -r_2 \frac{\partial}{\partial p_0} + 2p_2 \frac{\partial}{\partial q_0} \ ,$$

whose reduction on $\mathcal{M}_5$ is $Y = \partial/\partial h_4$. Since $Y(H(\lambda)) = 2h_2\lambda + 2h_4 - 2h_1h_2$ and $Y^2(H(\lambda)) = 2$, the coordinates μ_1 and μ_2 are the values of the polynomial

$$h_2\lambda + h_4 - h_1h_2$$

for $\lambda = \lambda_1, \lambda_2$. In order to check that the DN coordinates are separation variables for the restrictions $\widehat{H}_0$ and $\widehat{H}_1$ of the Hamiltonians to $\mathcal{S}_c$, we simply have to compute

$$\begin{bmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{bmatrix} \begin{bmatrix} \widehat{H}_0 \\ \widehat{H}_1 \end{bmatrix} = \begin{bmatrix} -c{\lambda_1}^2 + {\mu_1}^2 - {\lambda_1}^5 \\ -c{\lambda_2}^2 + {\mu_2}^2 - {\lambda_2}^5 \end{bmatrix},$$

from which the form of the spectral curve can also be seen.

9 Final Remarks

1. The results outlined in Section 7 are proved in [16] for a class of bi-Hamiltonian manifolds whose rank is not maximal. This means that our approach to the stationary reductions of KdV can be directly generalized to the stationary reductions of the Gel'fand–Dickey hierarchies. A step in this direction has already been taken in [17], whose results should be compared with those of [18, 31]. We will treat this problem in a future publication.
2. The separation variables provided by the bi-Hamiltonian method coincide, in the KdV case, with the ones obtained by algebro-geometric constructions. It would be interesting to compare in more general cases these two methods. A first result has been obtained in [20], where the "spectral Darboux coordinates" of [1] are shown to be DN coordinates for a suitable pair of compatible Poisson brackets.
3. Another Marsden-Ratiu reduction of the manifold $\mathcal{M}_A$ of Section 5 has been performed in [30], for an arbitrary simple Lie algebra $\mathfrak{g}$. That reduction leads to a bigger quotient space, and allows one to reduce all the multi–Hamiltonian structure of $\mathcal{M}_A$ and to obtain, in the case $\mathfrak{g} = \mathfrak{sl}(2)$, the Mumford systems [28]. A further restriction to the level surface of some Casimirs gives the same reduced phase space obtained in Section 6, where only two Poisson brackets survive.

Acknowledgments

J.P.Z. and M.P. were partially supported by FAPERJ through grant E-26/170.501/99-APV. J.P.Z is grateful to SISSA for its hospitality. We thank G. Tondo for useful discussions at the early stages of this work. G.F. wishes to thank B. Dubrovin for useful discussions and remarks. M.P. is grateful to IMPA and SISSA for their hospitality.

References

[1] M. R. Adams, J. Harnad, J. Hurtubise, *Darboux Coordinates and Liouville-Arnold Integration in Loop Algebras.* Commun. Math. Phys. **155** (1993), 385-413.

[2] M. Adler, P. van Moerbeke, *The complex geometry of the Kowalewski-Painlevé analysis.* Invent. Math. **97** (1989), 3-51.

[3] S. I. Alber, *On stationary problems for equations of Korteweg–de Vries type*, Comm. Pure Appl. Math. **XXXIV** (1981), 259–272.

[4] M. Antonowicz, A. P. Fordy, S. Wojciechowski, *Integrable stationary flows, Miura maps and Bi–Hamiltonian Structures*. Phys. Lett. A **124** (1987), 455–462.

[5] M. Błaszak, *On separability of bi–Hamiltonian chain with degenerated Poisson structures.* J. Math. Phys. **39** (1998), 3213–3235.

[6] O.I Bogoyavlensky, S. P. Novikov, *The relationship between hamiltonian formalism of stationary and non-stationary problems*. Funct. Anal. Appl. **10** (1976), 8–11.

[7] V.M. Buchstaber, V. Z. Enolskii, D.V. Leykin, *Hyperelliptic Kleinian function and applications.* In Solitons, Geometry and Topology: On the Crossroad (V.M. Buchstaber and S. P. Novikov, eds.), AMS Trans. Ser 2, Advances in Math. Sciences, Vol. **179**, 1997, pp. 1–34.

[8] P. Casati, G. Falqui, F. Magri, M. Pedroni, *A note on Fractional KdV Hierarchies*. J. Math. Phys. **38** (1997), 4605–4628.

[9] P. Casati, F. Magri, M. Pedroni, *Bi-Hamiltonian Manifolds and τ–function.* In: Mathematical Aspects of Classical Field Theory 1991 (M. J. Gotay et al. eds.), Contemporary Mathematics vol. **132**, American Mathematical Society, Providence, R. I., 1992, pp. 213–234,

[10] P. Casati, M. Pedroni, *Drinfeld–Sokolov Reduction on a Simple Lie Algebra from the Bi-Hamiltonian Point of View*. Lett. Math. Phys. **25** (1992), 89–101.

[11] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, *Transformation Groups for Soliton Equations.* Proceedings of R.I.M.S. Symposium on Nonlinear Integrable Systems–Classical Theory and Quantum Theory (M. Jimbo and T. Miwa, eds.), World Scientific, Singapore, 1983, pp. 39–119.

[12] L. A. Dickey, *Soliton Equations and Hamiltonian Systems.* Adv. Series in Math. Phys. Vol. **12**, World Scientific, Singapore, 1991.

[13] B. A. Dubrovin, I. M. Krichever, S. P. Novikov, *Integrable Systems I.* Enciclopædya of Mathematical Sciences, vol. **4** (Dynamical Systems IV), V. I. Arnol'd and S. P. Novikov, eds., Springer Verlag, Berlin, 1990.

[14] B. A. Dubrovin, V. B. Matveev, S. P. Novikov, *Nonlinear equations of KdV type, finite gap linear operators, and Abelian varieties.* Russ. Math. Surv. **31** (1976), 59–146.

[15] G. Falqui, F. Magri, M. Pedroni, *Bi-Hamiltonian Geometry, Darboux Coverings, and Linearization of the KP hierarchy*. Commun. Math. Phys. **197** (1998), 303–324.

[16] G. Falqui, F. Magri, M. Pedroni, in preparation.

[17] G. Falqui, F. Magri, G. Tondo, *Bi-Hamiltonian systems and separation of variables: an example from the Boussinesq hierarchy.* SISSA preprint 123/98/FM, to appear in Theor. Math. Phys.

[18] A.P. Fordy, S.D. Harris, *Hamiltonian Structures in Stationary Manifold Coordinates.* In: Algebraic Aspects of Integrable Systems (A.S. Fokas and I.M. Gelfand eds.), PND **26**, Birkhauser, Boston, 1996, pp. 103–130.

[19] I. M. Gel'fand, I. Zakharevich, *On the local geometry of a bi-Hamiltonian structure.* In: The Gelfand Mathematical Seminars 1990-1992 (L. Corwin et al. eds.), Birkäuser, Boston, 1993, pp. 51–112.

[20] J. Harnad, *R -matrices, bi-Hamiltonian systems, spectral invariants and separation of variables.* Preprint (1999).

[21] Y. Kosmann–Schwarzbach, F. Magri, *Poisson–Nijenhuis structures*. Ann. Inst. Poincaré (Phys. Theor.) **53** (1990), 35–81.

[22] I. M. Krichever, *Methods of algebraic geometry in the theory of non–linear equations.* Russ. Math. Surv. **32** (1977), 185–213.

[23] P. Lax, *Periodic Solutions of the KdV Equation.* Comm. Pure Appl. math. **XXVIII** (1975), 141-188.

[24] G. Magnano, F. Magri, *Poisson–Nijenhuis structures and Sato hierarchy.* Rev. Math. Phys. **3** (1991), 403–466.

[25] F. Magri, *Geometry and soliton equations.* In: *La Mécanique Analytique de Lagrange et son héritage*, Atti Acc. Sci. Torino Suppl. **124** (1990), 181–209.

[26] J.E. Marsden, T. Ratiu, *Reduction of Poisson Manifolds.* Lett. Math. Phys. **11** (1986), 161–169.

[27] C. Morosi, G. Tondo, *Quasi–Bi–Hamiltonian systems and separability.* J. Phys. A.: Math. Gen. **30** (1997), 2799-2806.

[28] D. Mumford, Tata Lectures on Theta II. Progr. in mathematics, Vol. **43**, Birkhäuser, Boston–Basel–Stuttgart, 1984.

[29] D. Mumford, P. van Moerbeke, *The spectrum of difference operators and algebraic curves.* Acta Math. **143** (1979), 93-154.

[30] M. Pedroni, P. Vanhaecke, *A Lie algebraic generalization of the Mumford System, its symmetries and its multi-Hamiltonian structure.* Regular and Chaotic Dynamics **3** (1998), 132–160.

[31] E. Previato, *Generalized Weierstrass $\wp$–functions and KP flows in affine space.* Comment. Math. Helvetici **62** (1987), 292–310.

[32] S. Rauch–Wojchechowski, *Newton representation for stationary flows of the KdV hierarchy.* Phys. Lett. **A 170** (1992), 91–94.

[33] A.G. Reyman, M.A. Semenov-Tian-Shansky, *Compatible Poisson structures for Lax equations: an r-matrix approach.* Phys. Lett. A **130** (1988), 456–460.

[34] M. Sato, Y. Sato, *Soliton equations as dynamical systems on infinite–dimensional Grassmann manifold.* In: Nonlinear PDEs in Applied Sciences (US-Japan Seminar, Tokyo), P. Lax and H. Fujita eds., North-Holland, Amsterdam, 1982, pp. 259–271.

[35] G. Segal, G. Wilson, *Loop Groups and equations of the KdV type.* Publ. Math. IHES **61** (1985), 5–65.

[36] E. Sklyanin, *Separations of variables: new trends.* Progr. Theor. Phys. Suppl. **118** (1995), 35–60.

[37] G. Tondo, *On the integrability of stationary and restricted flows of the KdV hierarchy.* J. Phys. A **28** (1995), 5097–5115.